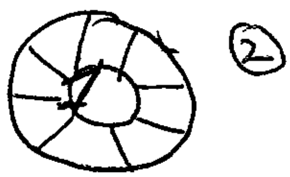
f.) Spherical Geometry

a.) Motivation:

→ $R = R(t)$, etc. ⟺ effects of imploding system, convergence etc.

→ Multiple Surfaces:



3 stages of R.T. instability:

a.) acceleration phase: — ablated material accelerated into shell ⇒ R.T. instability at ②

$R = 3mm \rightarrow 1.5mm$

b.) coasting phase: — shell coasts inward with no acceleration → ballistic ⟺ no acceleration ⇒ no R.T.

$R = 1.5mm \rightarrow 150\mu$

c.) decelleration phase: — shell decellerates, compressing gaseous core (gas ignitor) ⇒ R.T. instability at ① (g_{eff} outward)

$R = 150\mu \rightarrow 75\mu$

1 → inner

2 → outer

Impact on ICF design:

- naively, coasting phase appears benign, as no R.T. instability

- but a.) during acceleration, early coasting shell radius thin

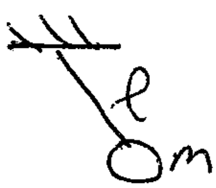
b.) $$\varphi\Big|_{R_1} = \exp\left[-\frac{\ell}{R_1}\left(R_2-\cancel{R_1}\right)\right]$$

⇒ outer surface R.T. seeds inner surface perturbation

⇒ inner surface perturbation grows due convergence prior to decceleration

- examples:

a.) Pendulum



with $\ell = \ell(t)$

$\dot{\ell}/\ell \ll \omega = \sqrt{g/\ell}$

Adiabatic invariant ⟷ action

$$S = \int p\,dq = \int p_\theta\,d\theta \quad ; \text{ here}$$

$$H = \frac{p_\theta^2}{2mL^2} + \frac{1}{2} mgL\theta^2 \quad \Rightarrow \quad p_\theta(\theta),\ \mathcal{S} \text{ etc.}$$

Alternatively: $\mathcal{S} \equiv$ Action
$\sim$ Energy $\times$ time

$$\mathcal{S} = \left(\tfrac{1}{2} mgLA^2\right)\left(\omega^{-1}\right)$$
$$= \frac{1}{2} m \frac{g}{L} L^2 A^2 \omega^{-1}$$
$$= \frac{1}{2} m \omega L^2 A^2$$

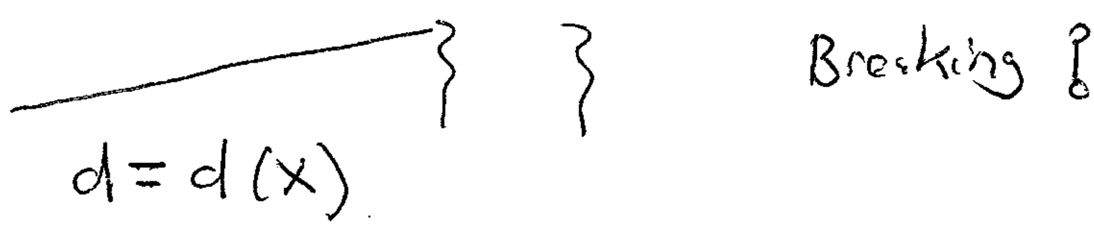
$$\therefore \quad \mathcal{S} \sim m L^{3/2} \sqrt{g}\, A^2$$

$$\Rightarrow \quad L_0^{3/2} A_0^2 = L(t)^{3/2} A(t)^2$$

$$A(t)/A_0 = \left(L_0/L(t)\right)^{3/4}$$

i.e. shortening string increases amplitude!

b.) Ocean Wave Impinging on Beach



Now, $\omega = \omega(k, g, d(x))$ for finite depth 'y analogy with pendulum:

Action Density (Wave) $N = \mathcal{E}/\omega$ $\mathcal{E}$ = wave energy density

[aside: QM: $E = (N + 1/2)\hbar\omega$

N → # quanta

Semiclassical: $E = N\hbar\omega$

Classical: $\hbar \to 1$, $N \to$ action $\Rightarrow N = \mathcal{E}/\omega$]

Then: - action density (= # waves) conserved along wave trajectories

- $$\frac{\partial}{\partial t} N + \underline{\nabla} \cdot (\underline{V}_{gr} N) = 0$$

$$\rightarrow V_{gr}(d(x)) \frac{\mathcal{E}(x)}{\omega(d(x))} = \text{const}$$

- Key Point:

- during coasting phase, inner surface is RT stable but

- supports surface waves, seeded by outer surface RT perturbations

- as surface wave ~ harmonic oscillator can expect growth as R shrinks during coasting

i.e.

Pendulum	Inner Surface Wave
$\omega = \sqrt{g/L}$	$\omega = \left(-\frac{\ell}{R_1}\ddot{R}_1\right)^{1/2}$
m	$M = \rho V$
	$= \rho\, 4\pi R_1^2 \Delta R_{pert.}$
	$\Delta R_{pert} = R_1/\ell = k^{-1}$
	$M = \rho\, 4\pi R_1^3/\ell$

$\therefore\ S = \frac{1}{2} m \omega L^2 A^2$

$$\rightarrow \frac{1}{2}\, 4\pi\rho \frac{R_1^3}{\ell}\left(-\frac{\ell}{R_1}\ddot{R}_1\right)^{1/2}\eta^2$$

$$= \rho \frac{R_1^3}{\ell}\,\omega\,\eta^2$$

$$\eta^2 \sim \frac{S_0 \, l}{\rho R_1^3 \omega} \sim \frac{\text{Const.}}{R_1^{5/2} (-\ddot{R}_1)^{1/2}}$$

$$\boxed{\eta \sim (-\ddot{R}_1)^{-1/4} R_1^{-5/4}}$$

$\Rightarrow$ perturbation grows by ($\times 10$) during coasting phase!

References: K. O. Mikaelian; Phys. Rev. A 42 3400
M. S. Plesset; J. Appl. Phys. 25 96 (1954)
(*) D. L. Book, S. E. Bodner, Phys. Fl. 30 367 (19[illegible]

b.) Analysis

i.) Coasting Shell "Equilibrium"

- Mass conserved during implosion

$$M = 4\pi \int_{R_1}^{R_2} \rho(R) R^2 dr$$

$\bar{\rho} \equiv$ avg. density

$$= \frac{4\pi}{3} \bar{\rho} (R_2^3 - R_1^3)$$

$$= (4\pi/3) \bar{\rho} R_0^3$$

($R_0 \equiv$ radius fully collapsed shell)

- shell incompressible (?!) – volume conserved

$$R^2 V = R^2 \dot{R} = R_1^2 \dot{R}_1$$

$$\dot{R} = \frac{R_1^2 \dot{R}_1}{R^2}$$

- For total energy: (K.E. of implosion)

$$W = \frac{1}{2} 4\pi \int_{R_1}^{R_2} \rho(R) R^2 dR \dot{R}^2$$

$$= 2\pi \int_{R_1}^{R_2} \bar{\rho} R^2 \frac{R_1^4}{R^4} \dot{R}_1^2 dR$$

$$W = 2\pi \bar{\rho} \dot{R}_1^2 R_1^3 (1 - R_1/R_2)$$

- time scale:

$$T_{implosion} = (M R_0^2 / 2W)^{1/2}$$ { mass, radius, energy

- For $\ddot{R}_1$ (i.e. dynamics of implosion)

$$\dot{W} = 0 = 2\pi\bar{\rho}\left[2 \dot{R}_1 \ddot{R}_1 R_1^3 (1 - R_1/R_2)\right.$$

$$+\ \dot{R}_1^{\,2}\left(3\dot{R}_1\right)\left(R_1^2\right)\left(1-R_1/R_2\right)$$
$$\left.+\ \dot{R}_1^{\,2}R_1^3\left(-\frac{\dot{R}_1}{R_2}+\frac{R_1\dot{R}_2}{R_2^2}\right)\right]$$

with $\quad R_2\dot{R}_2 = R_1^2\dot{R}_1$

(crank)

$\Rightarrow$
$$\ddot{R}_1 = -\frac{W}{4\pi\rho R_1^4}\left[3+2\frac{R_1}{R_2}+\left(\frac{R_1}{R_2}\right)^2\right]$$

observe for $\quad R_1 \ll R_2 \qquad$ (thick shell limit)

$$\ddot{R}_1 \sim -\frac{c}{R_1^4}$$

$$\dot{R}_1^{\,2} \sim \frac{c}{R_1^3} \quad \Rightarrow \quad dR\,R_1^{3/2} \sim dt$$

$$R_1^{5/2} \sim (t_0-t)$$

$$\Rightarrow \quad R_1 \sim (t_0-t)^{2/5}$$

$\downarrow$

implosion radius evolution.

ii.) Perturbations $\quad$ (RT/SW)

Take: $\quad$ - irrotational flow

- incompressible hydro. $\left(\ell c_s/R_1 \gg \left(-\frac{\ddot{R}_1}{R}\ell\right)\right)$

Then, as usual, have:

(i.) $\nabla^2 \phi = 0$; $\underline{V} = \nabla \phi$

(ii.) At interface $\Rightarrow$ Bernoulli Eqn.:

$$\rho \left(\frac{\partial \phi}{\partial t} + \frac{(\nabla \phi)^2}{2} \right) + p = 0$$

here, unperturbed V from imploding shell
$\Rightarrow$

$$\frac{\partial \tilde{\phi}}{\partial t} + \dot{R} \tilde{V}_r + \tilde{p} = 0 \qquad (1)$$

(iii.) Boundary conditions at interfaces:

$$\frac{d\tilde{\eta}_j}{dt} = \tilde{V}_{r_j} + \left(\frac{\partial V}{\partial R}\right)_j \tilde{\eta}_j$$

$\downarrow$
shell expanding here

$$\tilde{p}_j = - \left(\frac{\partial p}{\partial R}\right)_j \tilde{\eta}_j$$

$\downarrow$
shell expanding

but observe :

$$\rho \ddot{R} = -\frac{\partial P}{\partial R}$$

$$R^2 \dot{R} = R_j^2 \dot{R}_j = R^2 V$$

$$\therefore \left(\frac{\partial V}{\partial R}\right)_j = \left[\frac{R_j^2 \dot{R}_j}{(R+\Delta R)^2} - \frac{R_j^2 \dot{R}_j}{R^2}\right] \Big/ \Delta R$$

$$= -2 \frac{\dot{R}_j}{R_j}$$

$$\Rightarrow \boxed{\frac{d\tilde{\eta}_j}{dt} = \tilde{V}_{r_j} - 2\frac{\dot{R}_j}{R_j}\tilde{\eta}_j} \quad ②$$

similarly :

$$\boxed{\tilde{P}_j = \rho \ddot{R}_j \dot{\tilde{\eta}}_j} \quad ③$$

Now, to crank :

$$\nabla^2 \phi = 0 \Rightarrow$$

$$\phi(R,t) = \sum_{l,m} \left[R_1 V_1^{l,m}(t) \left(\frac{R_1}{R}\right)^{l+1} + R_2 V_2^{l,m}(t) \left(\frac{R}{R_2}\right)^{l}\right] * Y_{l,m}(\theta,\phi)$$

Then $\underline{V} = \underline{\nabla}\phi$

$$\underline{V}(R,t) = \sum_{\ell,m}\Big[V_1 (R_1/R)^{\ell+2}\big[-\hat{r}(\ell+1) Y_{\ell,m} + R\nabla Y_{\ell,m}\big] + V_2 (R/R_2)^{\ell-1} (\hat{r}\,\ell Y_{\ell,m} + R\nabla Y_{\ell,m})\Big]$$

substituting ϕ, $\nabla\phi$ into Bernoulli Eqn {ℓ^{th} mode

$$-\frac{P}{\rho} = Y_{\ell,m}\Big[\Big\{R_1\dot{V}_1 + (\ell+2)\dot{R}V_1 - (\ell+1)\dot{R}_1V_1\left(\frac{R_1}{R}\right) * \left(\frac{R_1}{R}\right)^{\ell+1} + \Big\{R_2\dot{V}_2 - (\ell-1)\dot{R}_2V_2 + \ell\dot{R}_2V_2(R_2/R)^3\Big\}\left(\frac{R}{R_2}\right)^{\ell}\Big]$$

Furthur, taking $\tilde{V}_r$ into Eqn. (2) to relate η, V_1, V_2 :

$$\dot{\eta}_1 + 2(\dot{R}_1/R_1)\eta_1 = -(\ell+1)V_1 + \ell A^{\ell-1}V_2$$

$$\dot{\eta}_2 + 2(\dot{R}_2/R_2)\eta_2 = -(\ell+1)A^{\ell+2}V_1 + \ell V_2$$

$A = R_1/R_2$

Similarly, plugging ③ into Bernoulli Eqn:

$$\left(R_1 \dot{V} + \left[R_2 \dot{V}_2 + (l A^{-3} - l + 1)\dot{R}_2 V_2\right]\right) A^{l} = -\ddot{R}_1 \eta_1$$

$$\left(R_1 \dot{V}_1 + \left[l+2 - (l+1) A^3\right]\dot{R}_1 V_1\right) A^{l+1} + (R_2 \dot{V}_2) = -\ddot{R}_2 \eta_2$$

Now:

→ during coasting phase, consider <u>thick</u> shell

$R_1 / R_2 = A \ll 1 \Rightarrow l, A \to \infty$

→ $\dot{\eta}_1 + 2(\dot{R}_1/R_1)\eta_1 = -(l+1) V_1$

$R_1 \dot{V}_1 = -\ddot{R}_1 \eta_1$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{1}{R_1}\frac{d}{dt}(R_1^2 \tilde{\eta}_1)\right) = (l+1) \ddot{R}_1 \tilde{\eta}_1}$$

Similarly

$$\dot{\eta}_2 + 2(\dot{R}_2/R_2)\,\eta_2 = \ell V_2$$

$$\dot{V}_2 = -\frac{\ddot{R}_2}{R_2}\,\eta_2$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{1}{R_2}\frac{d}{dt}\left(R_2^2\eta_2\right)\right) = -\ell\,\ddot{R}_2\,\eta_2}$$

$\Rightarrow$ two interfaces decouple!!

For inner interface (1):

$\rightarrow$ coasting shell $\quad R_1 \sim (t_0 - t)^{2/5}$

$$\eta_1 \sim (t_0 - t)^{\alpha} \quad \text{and}$$

$$\frac{d}{dt}\left(\frac{1}{R_1}\frac{d}{dt}\left(R_1^2\tilde{\eta}_1\right)\right) = (\ell+1)\,\ddot{R}_1\,\hat{\eta}_1$$

$$\Rightarrow \alpha = -\frac{1}{10} \pm (25 - 24\ell)^{1/2}$$

$$\therefore \quad \eta_1 \sim (t_0 - t)^{-1/10}$$

[slow increase due convergence]

note: consequence spherical convergence, not exponential growth.

> Wave-like solution (WKB)

$$\frac{d}{dt}\left(\frac{1}{R_1}\frac{d}{dt}\left(R_1^2\tilde{\eta}_1\right)\right) = (\ell+1)\ddot{R}\,\tilde{\eta}_1$$

$$\eta_1(t) = Z(t)\, e^{i\int^t \omega(t')dt'}$$

$\therefore$

$$-R_1\omega^2 Z + i\left(3\omega\dot{R}_1 Z + 2\omega R\dot{Z}_1 + \dot{\omega}R_1 Z\right)$$
$$+ R_1\ddot{Z} + 3\dot{R}\dot{Z} + 2\ddot{R}_1 Z = \; = (\ell+1)\ddot{R}Z$$

lowest order: $\omega^2 = -\frac{\ddot{R}_1}{R}(\ell+1)$

$\rightarrow$ eigen frequency

first order:

$$3\omega\dot{R}_1 Z + 2\omega R\dot{Z}_1 + \dot{\omega}R_1 Z = 0$$

$$\Rightarrow \quad R_1^3 Z^2 \omega = \text{const.}$$

Recovers adiabatic invariant !!
(SW Action, surface ①)

Implications follow!

Note:

i) Question significant to Nova Upgrade ⟷ advisability of long coasting phase

ii) Generally, for spherical R.T.:

$$\ddot{\eta} + 3\frac{\dot{R}}{R}\dot{\eta} - n\,A(n)\,\frac{\ddot{R}}{R}\,\eta = 0$$

$n \to \ell$

$$n\,A(n) = \left[n(n-1)\,\rho_2 - (n+1)(n+2)\,\rho_1\right] / \left[n\rho_2 + (n+1)\rho_1\right]$$

need: $\quad n\,A(n)\,\ddot{R} < 0$

$$\frac{d}{dt}\left[n\,A(n)\,R^5\ddot{R}\right] < 0$$

(Plasset)

(HW?)

1.) Nonlinear Rayleigh-Taylor Instability : Single Mode / Bubble

Ref.: D. Layzer Ap. J. 122 1 (1955). (a must
H. J. Kull, Review

A.) Motivation and Heuristics

Recall;

i.) in linear phase, simple R.T. instability

$$\gamma = \sqrt{gAk} \qquad (k\eta \ll 1)$$

ii.) in nonlinear phase expect algebraic growth (i.e. simple intuition) →  bubble rise $z(t)$

$$\ell = \alpha A g t^2 \qquad (k\eta \gtrsim 1)$$

{light rises via bubble}

Seek: → how recover algebraic growth?

→ how unify linear, nonlinear regimes?

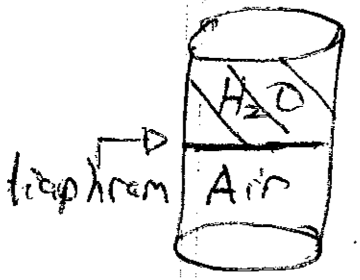
→ understand flow structure in nonlinear regime

Heuristically;

"NL": $\nabla\phi\cdot\nabla\eta \sim \frac{\partial\phi}{\partial z}$

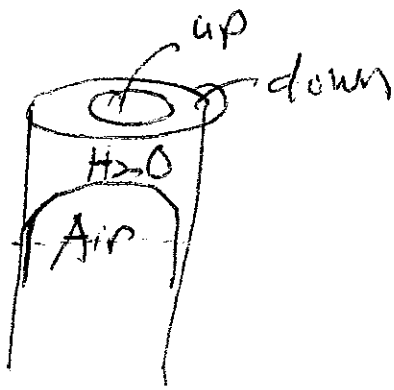
$\Rightarrow k\eta \sim 1$.

then, for cylinder:



$A = 1$

$\Rightarrow$



i.e. $\begin{cases} \text{air bubble rises in center} \\ H_2O \text{ spike falls at edge} \end{cases}$

$\Rightarrow$ "spike and bubble" picture R,T

∴ η (interface variable) $\rightarrow$ R (bubble radius)

(field, fourier modes) (structure)

then, linear theory $\Rightarrow$ $R \rightarrow \ell$

$$\gamma = \frac{1}{R}\frac{dR}{dt} = \sqrt{kg}$$

$$\frac{dR}{dt} = V = \sqrt{(kR)gR}$$

bubble rize velocity

$\therefore$ natural to delineate:

① $k_1 R = k\eta < 1$ — linear
$\Rightarrow R = R(0)\, e^{\gamma t}$

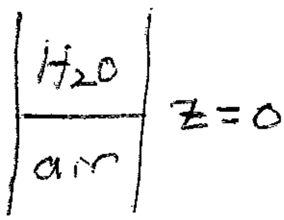
② $k_1 R \gtrsim 1$ $\quad V = \sqrt{gR}$ — non linear

i.e. $\quad \frac{dR}{dt} = \sqrt{gR} \Rightarrow R = \alpha g t^2$

Suggests can understand nonlinear R.T. via notions of bubble dynamics!

B.) D. Layzer Calculation

Consider tube:



Units: $\quad R/\beta_1 = 1$ (tube radius) $\qquad J_1(r)\big|_{r=\beta_1} = 0$ (first zero)

$g = 1$

$A = 1$

B.C's: Hard wall: $\quad V_r(z, \beta_1, t) = 0$

$\Rightarrow \partial_r \phi(z, \beta, t) = 0$

Evanescent at top: $V_z(z, r, t)\Big|_{z\to\infty} = 0$

$$\Rightarrow \quad \partial_z \phi(z, r, t)\Big|_{z\to\infty} = 0$$

Potential Flow: $\nabla^2\phi = 0$

Approach to solution: → How get/see 'spike and bubble' picture?
shape – streamlines

→ convert interface dynamics problem into (nonlinear) particle mechanics problem

→ use fluid particle eqns. of motion to determine stream-lines, equation for boundary

Now, general solution to $\begin{cases} \text{Laplace eqn.} \\ \text{B.C.'s} \end{cases}$

$$\phi = F(t)\, e^{-z} J_0(r)$$
↑
arbitrary fctn time

→ does not satisfy Bernoulli Eqn.

Then, for fluid particle:

$$\frac{dx}{\partial_x \phi} = \frac{dy}{\partial_y \phi} = \frac{dz}{\partial_z \phi}$$ → equations for streamlines

$$\frac{dr}{dt} = V_r = -\partial_r \phi$$

$$= + F(t) e^{-z} J_1(r)$$

$$\frac{dz}{dt} = V_z = -\partial_z \phi$$

$$= + F(t) e^{-z} J_0(r)$$

(Layzer notation)

exploits potential flow structure of problem $\rightarrow$ obviously not universally applicable

Then, for stream-lines, can write:

$$dz/dr = V_z / V_r$$

$$= (dz/dt)/(dr/dt)$$

$$= J_0(r) / J_1(r)$$

but Bessel identity $\Rightarrow$ $J_1'(r) = J_0(r) - \frac{J_1(r)}{r}$

$\therefore$

$$\frac{dz}{dr} = \frac{J_1'(r) + J_1(r)/r}{J_1(r)}$$

$$= \frac{J_1'}{J_1} + \frac{1}{r}$$

$$dz = \frac{J_1'}{J_1}\,dr + \frac{dr}{r}$$

$$z = \ln(J_1(r)) + \ln r$$

$$\Rightarrow \boxed{e^{z} = C\,r\,J_1(r)}$$

→ parametrizes fluid streamlines, if interface @ slightly distorted.
(no Bernoulli, yet)

Note:

- generate stream surfaces via displacement along $\hat{z}$
- stream surfaces like:

→ asymptotic to z axis

$\Rightarrow$ i.e. $e^{-|z|}\Big|_{z \to -\infty} \sim C\,r\,J_1(r)\Big|_{r \to R}$

$\Rightarrow$ streamline structure is under-pinning of spike + bubble intuition!

$\Rightarrow$ not really solutions, but artifact effect due to B.C.
i.e. generic! (see 249.)

Now, to obtain equation of motion for interface:

i.e.,

→ Solution not yet satisfy Bernoulli eqn $\Longleftrightarrow$ interface distortion

but

→ message is that B.C.'s force spike structure

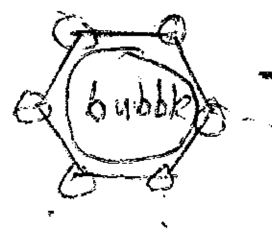
i.e. $$1 = \underbrace{c_n J_1(r)}_{\text{small}}\ \underbrace{e^{-z}}_{\text{large}}$$

$\therefore$

→ spike + bubble picture consequence $\left\{ \begin{array}{l} \text{B.C.'s} \\ \text{symmetry} \end{array} \right.$

i.e. hexagonal symmetry (cf. convection)

$\Rightarrow$



→ spikes at vertices
bubble at center

(top view):

→ integrate fluid equations of motion:

$$\frac{dr}{dt} = + F(t)\, e^{-z} J_1(r) \qquad \text{i.e.} \quad \frac{dx}{dt} = V = \nabla\phi$$

$$\frac{dz}{dt} = + F(t)\, e^{-z} J_0(r)$$

define: $Z = e^{z}$ → vertical variable

$V = r^2$ → radial

$T(t) = \int_0^t dt\, F(t) + 1$ → time variable

$k(V) = 2 J_1(r)/r$ → shape

Now,
$$\frac{dZ}{dt} = \frac{dz}{dt} e^{z} = + F(t)\, J_0(r) \qquad \text{(fluid eqns.)}$$
$$= + F(t)\, J_0(\sqrt{V})$$

$$\Rightarrow \boxed{\frac{dZ}{dT} = J_0(\sqrt{V})}$$

and

$$\boxed{Z = V k(V) / (dV/dT)}$$ → plus.

Eliminating Z :

$$\frac{V}{V_0} = (T-1).\, K(V_0) \,/\, Z_0 + 1$$ → velocity

$$\frac{Z}{Z_0} = \frac{K(V)}{K(V_0)}\left(\frac{(T-1)\,K(V_0)}{(Z_0+1)}\right)$$

Now:

→ potential can't solve Bernoulli eqn. over full surface

∴

→ seek expansion valid near bubble vertex, i.e. week distortion, see (26b.)

→ un-perturbed surface flat $\begin{cases} V_0 = 0 \\ Z_0 = 1 \end{cases}$

{flat surface approximation

$r \simeq 0$ cylindrical

∴ neglecting non-linearities in V ($r \simeq 0$)

↳ expansion

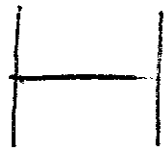
⇒ $$\frac{V}{V_0} = (T-1)\,\frac{K(V_0)}{Z_0+1}$$

$$= (T-1)\,\frac{2\cancel{K}}{2}$$

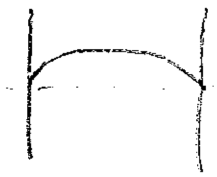
∴ $V = V_0\,(T-1)$

i.e.

→ unperturbed



→ weakly perturbed



→ strongly perturbed

$\underline{\Omega} \cong 0$ approx.

domain validity of treatment shrinks as bubble rises.

$\eta' \sim Z$

Similarly: $e^{\eta'} = (T-1)\left[1 - \frac{V}{8}(1-T^{-2})\right]$

$Z \sim \ln(T)$, $T \gg 1$

$(\eta' \sim Z)$

$\therefore \Rightarrow$ $\left\{ \begin{array}{l} V = V_0\,(T-1) \\ e^{\eta'} = (T-1)\left[1 - \frac{V}{8}(1-T^{-2})\right] \end{array} \right.$

Then: $\phi = F(t)\, e^{-Z}\, J_0(r)$ into Bernoulli

$\frac{\partial \phi}{\partial t} - (\nabla\phi)^2\ \eta \quad = 0$

$\Rightarrow \boxed{T\,(T^2+1)\,T'' - T'^2 - T^2\,(T^2-1) = 0}$

(coeff, V=0)

i.e. $\rightarrow F \rightarrow T'$

$\rightarrow \eta = \eta\,(T, V)$ (above)

$\rightarrow\ ' = \frac{d}{dt}$ (Bernoulli in real time)

Check:

a.) Linear Regime (small time)

$T = 1 + \tau$, $\tau \ll 1$

$\Rightarrow (1+\tau)\left((1+\tau)^2+1\right)\tau'' - \tau'^2 - (1+\tau)^2\left((1+\tau)^2-1\right) = 0$

$$\therefore \quad T'' - 2T = 0$$

i.e. $T = T(0)\, e^{t}$ → exponential growth a la' linear theory. ✓

In dimensional units:

$$T = T(0)\, e^{\gamma t} \qquad \gamma = \sqrt{\frac{Rg}{\beta_1}}$$

$$\sim \sqrt{k_{eff}\, g}$$

↳ set by cylindrical geometry

b.) Nonlinear Regime $\quad T \gg 1$

$$T^3 T'' - \cancel{T'^2} - T^4 = 0$$

$$T'' - T = 0$$

$$\Rightarrow \quad T = e^{t} \quad \Rightarrow \quad T' = F(t) = e^{t}$$

$$\underset{\text{Bubble rise}}{V} = \left.\frac{d\eta}{dt}\right|_{r=0} = \frac{d}{dt} \frac{\ln(T)}{T}$$

$$= \frac{1}{T}\frac{dT}{dt} = 1 \checkmark$$

(interface eqn.)

$$\left[\begin{array}{l} e^{\eta'} = (T-1) \quad (r \to 0) \\ \eta' e^{\eta'} = \dot{T} \quad \eta \sim \ln T \\ T \gg 1 \end{array}\right.$$

Then $V = 1$, or in dimensional units:

$$\boxed{V = \left(\frac{gR}{B_1}\right)^{1/2}}$$ Agrees with free-fall intuition

c.) Can find general vortex dynamics solution.

d.) $\left.\begin{array}{l} T \sim 1 \\ T \gg 1 \end{array}\right\}$ limits establish $k\eta \sim 1$ as criterion for entrance into nonlinear regime.

C.) Heuristics for Multiple Mode Systems

- Layzer solution for single bubble/mode
- in reality, ICF target finnish irregularities initialize many modes

Seek: multi-mode criterion for non-linearity
⇒ amplitude for exponential growth cessation ?!

ref: S. Haan, Phys. Rev. A **39** 5812 (1989).

First, observe: all modes can't grow to $k\eta_{\underline{k}} \sim 1$
$\Rightarrow$ spectral content diverges!

i.e.

$$\langle \xi^2 \rangle = \int d\underline{k}\, |k\tilde{\eta}_{\underline{k}}|^2$$

$$= \int k\, dk\, \frac{1}{k^2}$$

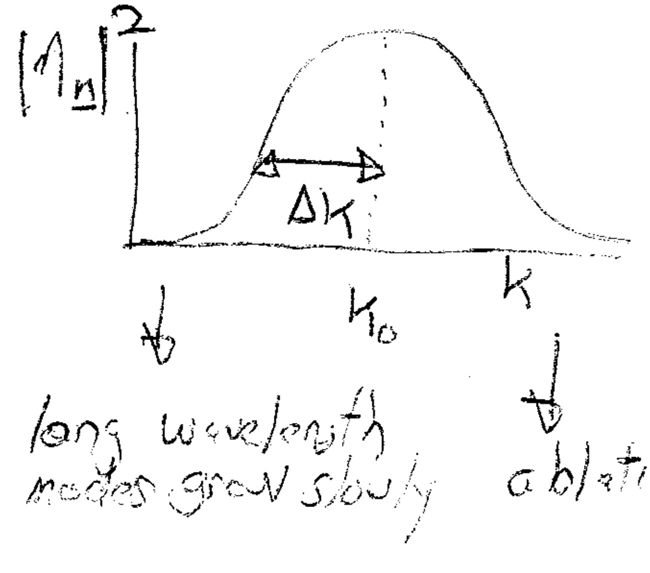
$$\rightarrow \infty$$

$\therefore$ Some modes, especially long wavelength, remain in linear regime (slow)

$\therefore$ natural, in multi-mode case, to suggest nonlinearity criterion $\underbrace{k_0 \eta_{rms}}$ ~ 1, where k_0 refers to wave number of spectral peak

from superposition modes

i.e.



{ spectrum $\rightarrow$ peak + width

$$\therefore \quad k_0^{-2} = \frac{L^2}{(2\pi)^2}\int_0^\infty 2\pi k\, dk\, |\tilde{\eta}_k|^2 = \langle \tilde{\eta}^2 \rangle_{ms}$$

$$= \frac{L^2}{2\pi} k_0 \Delta k\, |\tilde{\eta}_{k_0}|^2$$

$\Rightarrow$ establishes criterion:

$$\eta_{k_0} \cong \frac{\sqrt{2\pi}}{L}\left(k_0^3 \Delta k\right)^{-1/2}$$

if $\Delta k \sim k_0$ (frequent state affairs)

$\Rightarrow$

$$\boxed{\eta_{k_0} \cong \frac{\sqrt{2\pi}}{L k_0^{+2}}}$$

scaling with system!

contrast to $\eta_{k_0} \sim 1/k_0$

$\Rightarrow$

$\rightarrow$ in multimode system, superposition (bubble competition) of many-mode interface displacements $\Rightarrow k_0 \eta_{k_0} \cong \frac{\lambda_0}{L}$, not 1 (a la single wave)

$\rightarrow$ transition to Layzer regime at lower amplitude.

$\rightarrow$ consistent with LLNL simulations, experiments

e) Bubble Competition and Mix $\leftrightarrow$ Heuristics

$\rightarrow$ goal is to:

- construct model of nonlinear Rayleigh-Taylor using Layzer spike + bubble model

- generate model of growth of mixing layer

$\rightarrow$ ingredients:

- single bubble model

$\Downarrow$

- bubble-bubble interaction $\Rightarrow$ Competition

A.) Single Bubble

- Layzer calculation $\Rightarrow$ bubble vertex rises at $V = (gR/\beta_1)^{1/2}$

$\therefore$ suggests that mixing layer grows as:

$$\frac{dl}{dt} = (gl)^{1/2} \qquad l \sim \alpha g t^2$$

but

omits other effects which limit bubble / spike dynamics:

→ compression

→ drag, spike KH

1 D Model:
$$\begin{cases} V(\ell, z, t) \rightarrow \text{bubble rise velocity } (z \\ \ell(z, t) \rightarrow \text{mixing layer width} \end{cases}$$

$$\frac{\partial \ell}{\partial t} + \underbrace{V \frac{\partial \ell}{\partial z}}_{\text{bubble compression}} = \underbrace{V}_{\text{bubble rise velocity } \sim (g\ell)^{1/2}}$$

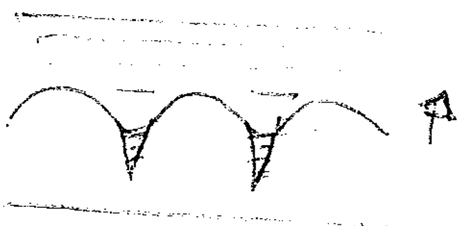
$$\rho \frac{\partial V}{\partial t} = \underbrace{\rho g}_{\text{gravitational force}} - \rho C_D \frac{V^2}{\ell} \rightarrow \underline{\text{drag}}$$

(D. Youngs
Physica D 37 (198
270
+ refs therein

Phyics of Drag:

- view bubble rise as mixture of two inter-penetrating fluids:

- i.e.



$\begin{cases}\text{inter-penetrating} \\ \text{fluids}\end{cases}$

$$F_D = \overbrace{C_D}^{\text{drag coefficient (phenomenological)}} \underbrace{\frac{V}{\ell}}_{\text{time scale}} \underbrace{\rho V}_{\text{momentum}}$$

($\ell \Leftrightarrow$ "chunk" size for interpenetrating fluids)

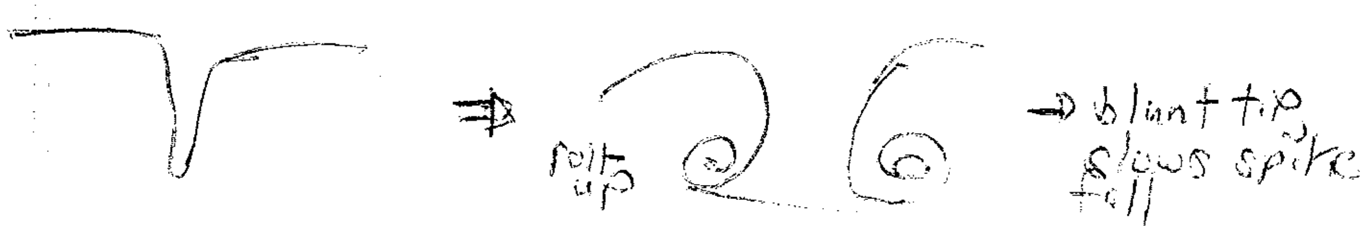
Drag slows bubble rise

i.e. $\frac{\partial V}{\partial t} = 0 \Rightarrow V = \left(\frac{1}{C_D}\right)^{1/2} (g\ell)^{1/2}$

$$\ell = C_D^{-1/2} \frac{g}{2} t^2$$

$\therefore$ Drag coefficient contains physics of $\alpha \sim .05$ scaling!

- alternatively, view drag term as manifestation of decay due to spike shear flow instability
i.e.:
recall:



Now time scale for roll-up is flow shear rate,

i.e. $\gamma \sim |\nabla u|$

$\therefore$ dimensionally, $\gamma \sim \frac{V}{\ell}$ $\Rightarrow$ rate of drag.

$\Rightarrow$ Drag $\left\{\begin{matrix}\text{term} \\ \text{coefficient}\end{matrix}\right\}$ essential to proper fit of implosion experiment data!

B.) Bubble Competition

observe:

- single bubble mixing model characterized by single length scale

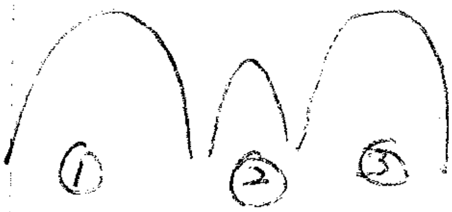
but

- multi-mode system $\Rightarrow$ multi-bubbles

$\therefore$

- need describe trend in evolution of bubble length scale

Simple example:

→ 

$\underline{t}$

3 bubbles
exterior larger

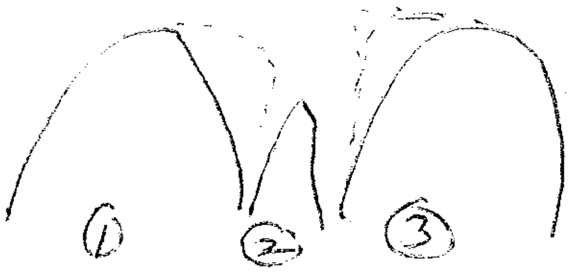
$\ell_{2} < \ell_{1}, \ell_{3}$

→ but, in Layzer regime:

$$V \sim (g\ell)^{1/2}$$

⇒ larger bubbles rise faster

→ ∴, at $t+\Delta t$



⇒ 1, 3 will tend to expand over 2, thus "squeezing" it out

$\therefore$

→ trend in bubble competition is:

- larger bubbles rise faster
- squeeze out smaller bubbles

$\therefore$

⇒ trend is progression towards single length scale characteristic of largest bubble
⇒ seen as "inverse cascade".

→ can build multi-bubble Layzer model (2D with discrete dynamics in z).

Physics 218A

P. Diamond

Bubble, Bubble, Toil and Trouble ...

Last time:

→ discussed D. Layzer('55) solution for single, nonlinear bubble.

- why: observables in NL state

$\gamma'(1\text{cm}) \sim .1$ sec.

- what: → single mode ($\lambda \leftrightarrow R$) growth in $\left\{ \begin{array}{l} k\eta \sim 1 \ (\eta/R \sim 1) \\ t \to \infty \end{array} \right.$ regime

i.e. $V = \#(gR)^{1/2}$

→ connects exponential and algebraic growth regimes (NL saturation).

- how: → flat interface approximation at (i.e. $\Gamma \to 0$) $\leftrightarrow$ geometry bubble tip.

→ use of streamlines from $\nabla^2\phi = 0$ and (assumed) self-similarity (valid at bubble tip).

→ Today:
(i.) Lazy Man's Layzer
(ii.) Bubble Competition

I.) Recall basic R-T equations:

- $\nabla^2\phi = 0$

- $\dfrac{\partial \eta}{\partial t} + \underline{\nabla}\phi \cdot \underline{\nabla}\eta = V_z$ $\underline{\underline{\text{or}}}$, more generally, "interface moves with fluid"

- $\dfrac{\partial \phi}{\partial t} + \underline{\nabla}\phi \cdot \underline{\nabla}\phi + g\eta = \text{const.}$

For single mode (bubble):

$$\phi(x,z,t) = a(t)\cos(kx)\,e^{-kz}$$

→ since bubble is rising light fluid, bubble flow must vanish at $+\infty$.

Now, essence of Layzer / bubble tip approximation is geometric ↔ assumed shape of bubble, i.e.:

$$z_i(x,t) = \eta(x,t) = z_0 + z_1\,(x-x_i)^2$$

i.e. parabolic shape approximation:

X_i, Z_0 → tip location
Z_1 → radius of curvature of bubble
i.e. $R_c = -1/2Z_1$

Note: Linear: $\gamma \eta = \frac{\partial \phi}{\partial z}$
i.e. ϕ forces η
Layzer: $\eta \leftrightarrow$ geometry

Then:

$$\eta = z = Z_0 + Z_1 (x - X_1)^2$$

so
$$\frac{\partial \eta}{\partial t} + \underline{V} \cdot \underline{\nabla} \eta = \frac{\partial \phi}{\partial z}$$

$$\Rightarrow \frac{dZ_0}{dt} + (x - X_1)^2 \frac{dZ_1}{dt} + (-2) Z_1 (x - X_1) \frac{dX_1}{dt}$$

$$+ V_x \, 2 (x - X_1) = V_z$$

$$\Rightarrow \qquad V_z - \frac{dz_0}{dt} - \frac{dz_1}{dt}(x-x_1)^2$$

$$- 2z_1 (x-x_1)\left(V_x - \frac{dx_1}{dt}\right) = 0$$

$\therefore$ have:

$$\underset{①}{\phi} = \underset{②}{a(t)}\cos(\underset{③}{kx})\, e^{-kz} \quad ; \quad \underline{V} = \underline{\nabla}\phi$$

$$\left.\begin{aligned} &V_z - \frac{dz_0}{dt} - \underset{④}{\frac{dz_1}{dt}}(x-x_1)^2 \\ &= -2z_1(x-x_1)\left(V_x - \frac{dx_1}{dt}\right) \end{aligned}\right\} \text{interface eqn.}$$

and

$$\frac{\partial\phi}{\partial t} + \frac{V_x^2 + V_z^2}{2} + gz = \text{const.}$$

Plugging in, to 2^{nd} order:

Interface eqn: $O(a)$;

$$-ka(t)\underbrace{(\cos kx)}_{1}\Big|_{\substack{x=0\\(\text{t.p})}} e^{-kz_0} - \frac{dz_0}{dt} = 0$$

$$\Rightarrow \boxed{a k e^{-kz_0} + \frac{dz_0}{dt} = 0}$$

to $O(2)$:

$$\boxed{a k^2\left(z_1 + \frac{k}{2}\right) e^{-kz_0} - \frac{dz_1}{dt} + 2 z_1 a k^2 e^{-kz_0} = 0}$$

and Bernoulli (to $O(2)$):

$$\boxed{\underset{\partial\phi/\partial t}{k e^{-kz_0}\left(z_1 + \frac{k}{2}\right)\frac{da}{dt}} + \underset{v^2}{a^2 k^3 z_1 e^{-2kz_0}} - g z_1 = 0}$$

Recall: $\rightarrow e^{-kz}$ $\quad$ HW.

$$z \rightarrow z_0 + z_1 (x - x_i^2)$$

$\rightarrow$ bubble tip $(x \rightarrow 0)$

$\Rightarrow$ equiv. Layzer Solution.

Check: Linear solution:

$$\frac{k^2}{2} e^{-kz_0} \frac{da}{dt} = g z_1$$

$$\frac{a k^3}{2} e^{-kz_0} = \frac{dz_1}{dt}$$

so, small perturbations $\Rightarrow$

$$\frac{k^2}{2} e^{-kz_0} \frac{d^2a}{dt^2} = \frac{g a k^3}{2}$$

$$\frac{d^2a}{dt^2} = g k a \quad \checkmark \text{ etc.}$$

Now, for late times:

$$k e^{-kz_0}\left(z_1 + \frac{k}{2}\right) \cancel{\frac{da}{dt}} + a^2 k^3 z_1 e^{-2kz_0} = g z_1$$

mode sats.

$\left\{ \begin{array}{l} v^2 \sim g\eta \sim g\lambda \\ \text{balance} \end{array} \right.$

$\Rightarrow a^2 k^3 \sim g$

$(ka)^2 \sim g/k \Rightarrow v^2 \sim g\lambda$

akin $v^2 \sim (gR)$

etc.

→ what's important?

① → birth/death structure
② → self-similarity (i.e. all N-coagulates same)
③ → merger rules.

① – ③ ⇒ inverse cascade.

More general form:

$$N(t)\frac{\partial g(\lambda,t)}{\partial t} = \overbrace{-2g(\lambda,t)\int_0^\infty g(\lambda',t)\,w(\lambda,\lambda')\,d\lambda'}^{\text{death}} + \underbrace{\int_0^\lambda g(\lambda-\lambda',t)\,g(\lambda',t)\,w(\lambda-\lambda',\lambda)\,d\lambda'}_{\text{birth}}$$

$N = \int_0^\infty g(\lambda,t)\,d\lambda \equiv$ # bubbles at t

$g(\lambda,t)\,d\lambda$ = size distribution function

$w(\lambda_i,\lambda_{i+1},t)$ → merger rate $\quad \lambda_i,\ \lambda_{i+1} \to \lambda_i + \lambda_{i+1}$